

# Groups of components of Néron models of Jacobians and Brauer groups

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**ABSTRACT.** Let  $X$  be a proper, smooth, and geometrically connected curve over a non-archimedean local field  $K$ . In this paper, we relate the component group of the Néron model of the Jacobian of  $X$  to the Brauer group of  $X$ .

## 1. INTRODUCTION

Let  $K$  be a non-archimedean local field. Thus  $K$  is a complete discrete valuation field with finite residue field  $k$ . Let  $K^{\text{nr}}$  be the maximal unramified extension of  $K$ . Let  $X$  be a proper, smooth, and geometrically connected curve over  $K$ , and  $X^{\text{nr}} = X \otimes_K K^{\text{nr}}$  be the corresponding curve over  $K^{\text{nr}}$ . Let  $\delta$  and  $\delta'$  denote, respectively, the *index* and the *period* of  $X$ , and let  $\delta^{\text{nr}}$  and  $\delta^{\text{nr}'}$  denote the corresponding quantities associated to  $X^{\text{nr}}$ . Let  $A$  be the *Jacobian variety* of  $X$  over  $K$ ,  $\Phi_A$  the  $k$ -group scheme of connected components of the Néron model of  $A$ , and  $c_A = \#\Phi_A(k)$  the corresponding *Tamagawa number* of  $A$  at  $K$ . Consider the Brauer-Grothendieck group  $\text{Br}(X) = H^2(X, \mathbb{G}_m)$ . Let  $\text{Br}_0(X)$  denote the image of  $\text{Br}(K) \rightarrow \text{Br}(X)$ , and  $\text{Br}_{\text{nr}}(X)$  denote the kernel of  $\text{Br}(X) \rightarrow \text{Br}(X^{\text{nr}})$ . In this paper, we prove

**Theorem 1.1** (Main Theorem). *There exists an exact sequence*

$$0 \rightarrow \text{Hom}(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z}) \rightarrow \Phi_A(k) \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow 0$$

where  $d = \delta'/\delta^{\text{nr}'}$ .

It follows that

**Corollary 1.2.**  *$\text{Br}_{\text{nr}}(X)/\text{Br}_0(X)$  is a finite group of order  $c_A/d$ .*

Corollary 1.2 has an interesting application. To explain this, consider a *global* field  $K$ . Thus  $K$  is either a finite extension of  $\mathbb{Q}$  i.e. a number field, or is finitely generated and of transcendence degree 1 over a finite field  $k$  i.e. a function field. In the number field case, let  $U$  denote a nonempty open subscheme of  $\text{Spec } \mathcal{O}_K$ , and in the function field case, let  $U$  denote a nonempty open subscheme of the unique, smooth, complete, and irreducible curve  $V$  over  $k$  whose function field

is  $K$ . Consider a regular, connected scheme  $\mathcal{X}$  of dimension 2 with a proper morphism  $\pi : \mathcal{X} \rightarrow U$  such that its generic fiber  $X = \mathcal{X} \otimes_U K$  is a smooth and geometrically connected curve over  $K$ . Let  $S$  be the set of primes of  $K$  not corresponding to a point of  $U$ , and  $\overline{K}$  be the separable algebraic closure of  $K$ . Note that  $S$  contains all archimedean primes of  $K$  in the number field case, and that it may be empty in the function field case. For each prime  $v \notin S$ , let  $K_v$  denote the completion of  $K$  at  $v$ , and let  $X_v = X \otimes_K K_v$ . Let  $\delta$  and  $\delta'$  be, respectively the index and period of  $X$  while  $\delta_v$  and  $\delta'_v$  be the corresponding quantities associated to  $X_v$ . It is known that  $\delta_v \neq 1$  for only finitely many primes  $v$ , and that either  $\delta_v = \delta'_v$  or  $\delta_v = 2\delta'_v$  [Lic69, Theorem 8]. Let  $\text{Br}(\mathcal{X})$  denote the Brauer group of  $\mathcal{X}$  and define  $\text{Br}(\mathcal{X})'$  by the exactness of the sequence

$$0 \rightarrow \text{Br}(\mathcal{X})' \rightarrow \text{Br}(\mathcal{X}) \rightarrow \bigoplus_{v \in S} \text{Br}(X_v)$$

Now let  $A/K$  be the Jacobian variety of  $X$  over  $K$ , and denote by  $\text{III}(A/K)$  the *Shafarevich-Tate group* of  $A/K$ . Generalizing the work of Artin [Tat68] and Milne [Mil82], Gonzalez-Aviles has shown [Gon03] that

**Theorem 1.3** (Gonzalez-Aviles). *Suppose that the integers  $\delta'_v$  are pairwise co-prime and that  $\text{III}(A/K)$  contains no nonzero infinitely divisible elements. Then there is an exact sequence*

$$0 \rightarrow T_0 \rightarrow T_1 \rightarrow \text{Br}(\mathcal{X})' \rightarrow \text{III}(A/K)/T_2 \rightarrow T_3 \rightarrow 0$$

in which  $T_0, T_1, T_2$  and  $T_3$  are finite groups of orders

$$\begin{aligned} \#T_0 &= \delta/\delta' \\ \#T_1 &= 2^e \\ \#T_2 &= \delta'/\prod \delta'_v \\ \#T_3 &= \frac{\delta'/\prod \delta'_v}{2^f} \end{aligned}$$

where

$$e = \max(0, d' - 1)$$

and

$$f = \begin{cases} 1 & \text{if } \delta'/\prod \delta'_v \text{ is even and } d' \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here  $d'$  is the number of primes  $v$  for which  $\delta_v = 2\delta'_v$ . In particular, if one of  $\text{III}(A/K)$  or  $\text{Br}(\mathcal{X})'$  is finite, then so is the other, and their

orders are related by

$$\delta\delta' \# \mathrm{Br}(\mathcal{X})' = 2^{e+f} \prod_v (\delta_v')^2 \# \mathrm{III}(A/K)$$

Now let  $K_v^{\mathrm{nr}}$  be the maximal unramified extension of  $K_v$ , and  $X_v^{\mathrm{nr}} = X_v \otimes_{K_v} K_v^{\mathrm{nr}}$  be the fiber over  $K_v^{\mathrm{nr}}$ , with index  $\delta_v^{\mathrm{nr}}$  and period  $\delta_v^{\mathrm{nr}'}$ . Let  $\mathrm{Br}_0(X_v)$  be the image of the map  $\mathrm{Br}(K_v) \rightarrow \mathrm{Br}(X_v)$ , and  $\mathrm{Br}_{\mathrm{nr}}(X_v)$  be the kernel of the map  $\mathrm{Br}(X_v) \rightarrow \mathrm{Br}(X_v^{\mathrm{nr}})$ . Let  $c_{A,v}$  the Tamagawa number of  $A$  at  $v$ . Note that Theorem 1.1 applies to  $X_v$ . Combining Corollary 1.2 with Theorem 1.3, we obtain

**Corollary 1.4.** *Suppose that the integers  $\delta_v'$  are pairwise co-prime, and  $\mathrm{III}(A/K)$  is finite. Then*

$$\# \mathrm{III}(A/K) \prod_v c_{A,v} = M \# \mathrm{Br}(\mathcal{X})' \prod_v \#(\mathrm{Br}_{\mathrm{nr}}(X_v)/\mathrm{Br}_0(X_v))$$

where  $M$  is a rational number given by

$$M = \frac{\delta\delta'}{2^{e+f} \prod_v \delta_v' \delta_v^{\mathrm{nr}'}}$$

Of course, the left-hand term in the above formula appears in the statement of the well-known Birch and Swinnerton-Dyer Conjecture.

*Remark 1.5.* The hypothesis that the integers  $\delta_v'$  are pairwise coprime in Theorem 1.3 can be dropped when  $K$  is a function field, and  $S = \emptyset$ . More precisely, assume that the curve  $V/k$  introduced above is also geometrically connected, and consider  $\mathcal{X}$  to be a smooth, proper, and geometrically connected surface endowed with a proper and flat morphism  $f : \mathcal{X} \rightarrow V$  whose generic fiber is  $X \rightarrow \mathrm{Spec} K$ . In this case, it is shown in [LLR05, Cor. 3] that if, for some prime  $l$ , the  $l$ -part of the group  $\mathrm{Br}(\mathcal{X})$  or of the group  $\mathrm{III}(A/K)$  is finite, then

$$\delta^2 \# \mathrm{Br}(\mathcal{X}) = \prod_v \delta_v \delta_v' \# \mathrm{III}(A/K)$$

and  $\# \mathrm{Br}(\mathcal{X})$  is a square. It follows that, in this case, we get

$$\# \mathrm{III}(A/K) \prod_v c_{A,v} = N \# \mathrm{Br}(\mathcal{X}) \prod_v \#(\mathrm{Br}_{\mathrm{nr}}(X_v)/\mathrm{Br}_0(X_v))$$

where the rational number  $N$  is given by

$$N = \frac{\delta^2}{\prod_v \delta_v \delta_v^{\mathrm{nr}'}}$$

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## 2. PRELIMINARIES

**2.1. Component Groups, Tamagawa numbers.** Let  $K$  be a complete, discretely valued field with finite residue field  $k$ . Let  $A/K$  be an abelian variety over  $K$ , and let  $\mathcal{A}$  be the *Néron model* [BLR90] of  $A/K$  over  $\mathrm{Spec} \mathcal{O}_K$ . The closed fiber  $\mathcal{A}_k$  of  $\mathcal{A}$  is a  $k$ -group scheme, not necessarily connected. Let  $\mathcal{A}_k^0$  be the connected component of  $\mathcal{A}_k$  containing the identity. Over  $\mathrm{Spec} k$ , there is an exact sequence of group schemes

$$0 \rightarrow \mathcal{A}_k^0 \rightarrow \mathcal{A}_k \rightarrow \Phi_A \rightarrow 0$$

where the quotient  $\Phi_A$  is a finite, étale group scheme over  $k$ . Equivalently,  $\Phi_A$  is a finite abelian group with a continuous action of  $\mathrm{Gal}(\bar{k}/k)$  on it. The group scheme  $\Phi_A = \mathcal{A}_k/\mathcal{A}_k^0$  is called the *component group* of  $A$ . The group of rational points  $\Phi_A(k)$ , called the *arithmetic component group* of  $A$ , counts the number of connected components of  $\mathcal{A}_k$  which are geometrically connected and  $c_A = \#\Phi_A(k)$  is called the *Tamagawa number* of  $A/K$ . Now let  $K^{\mathrm{nr}}$  be the maximal unramified extension of  $K$ . The inclusion  $\mathrm{Gal}(\bar{K}/K^{\mathrm{nr}}) \subset \mathrm{Gal}(\bar{K}/K)$  induces a map  $H^1(K, A) \rightarrow H^1(K^{\mathrm{nr}}, A)$ , whose kernel corresponds to the *unramified* subgroup of  $H^1(K, A)$ . The map may also be given as  $WC(A/K) \rightarrow WC(A/K^{\mathrm{nr}})$  where,  $WC(A/K) \cong H^1(K, A)$  denotes the Weil-Châtelet group of  $A$  over  $K$ . We denote this kernel by  $\mathrm{TT}(A/K)$  and call it the group of *Tamagawa torsors* of  $A$  over  $K$  [Bis13].

**Theorem 2.1.** *There exists a canonical isomorphism of finite abelian groups*

$$\mathrm{TT}(A/K) \cong H^1(k, \Phi_A)$$

*Proof.* The inflation-restriction sequence

$$0 \longrightarrow H^1(K^{\mathrm{nr}}/K, A(K^{\mathrm{nr}})) \longrightarrow H^1(K, A) \longrightarrow H^1(K^{\mathrm{nr}}, A)$$

identifies the set of Tamagawa torsors with the injective image of the group  $H^1(K^{\mathrm{nr}}/K, A(K^{\mathrm{nr}}))$  in  $H^1(K, A)$ . There is an isomorphism [Mil86, §Prop I.3.8]  $H^1(K^{\mathrm{nr}}/K, A(K^{\mathrm{nr}})) \cong H^1(k, \Phi_A)$  and the latter group is finite, since  $\Phi_A$  is finite.  $\square$

**Corollary 2.2.** *Suppose that  $A/K$  is a Jacobian variety. Then there exists a canonical perfect pairing of finite abelian groups*

$$\mathrm{TT}(A/K) \times \Phi_A(k) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

*In particular,  $\mathrm{TT}(A/K)$  has order  $c_A$ .*

*Proof.* This follows from Theorem 2.1, the fact that  $A$  is a self-dual abelian variety, and the perfectness of the pairing [McC86, (4.5)] induced by Grothendieck's pairing [McC86, (2.1)].  $\square$

Thus, for any Jacobian variety  $A/K$ , there is an isomorphism

$$(2.1) \quad \Phi_A(k) \cong \mathrm{Hom}(\mathrm{TT}(A/K), \mathbf{Q}/\mathbf{Z})$$

of finite, abelian groups.

*Remark 2.3.* Since  $\Phi_A$  is finite, its Herbrand quotient is 1 which implies that  $\#H^1(k, \Phi_A) = \#H^0(k, \Phi_A) = c_A$ . Thus, for any abelian variety  $A/K$  (and not just Jacobians), it follows from Theorem 2.1 that  $\#\mathrm{TT}(A/K) = c_A$ .

## 2.2. Picard Groups, Jacobian Varieties, and Brauer Groups.

Let  $X$  be a smooth, projective, geometrically connected curve defined over any field  $K$ . Let  $\overline{K}$  be a separable closure of  $K$ , and let  $\overline{X} = X \otimes_K \overline{K}$ . Let  $\mathrm{Div}(\overline{X})$  be the group of divisors of  $\overline{X}$  i.e. the free abelian group generated by the points of  $X(\overline{K})$ . Note that  $\mathrm{Div}(\overline{X})^{G_K} = \mathrm{Div}(X)$ , where  $G_K = \mathrm{Gal}(\overline{K}/K)$ . There is a natural summation map  $\mathrm{Div}(X_K) \rightarrow \mathbf{Z}$  whose image is  $\delta\mathbf{Z}$ , where  $\delta$  is the *index* of  $X$ . Equivalently,  $\delta$  is the least positive degree of a divisor in  $\mathrm{Div}(X_K)$ . Let  $P = \mathrm{Pic}_X$  be the *Picard scheme* of  $X$  so that  $P(\overline{K}) = \mathrm{Pic}(\overline{X})$ . It follows that  $P(K) = P(\overline{K})^{G_K} = \mathrm{Pic}(\overline{X})^{G_K}$ . The Picard scheme is a smooth group scheme over  $K$  whose identity component  $A = \mathrm{Pic}_X^0$  is called the *Jacobian variety* of  $X$ . There is an exact sequence of  $G_K$ -modules

$$(2.2) \quad 0 \rightarrow A(\overline{K}) \rightarrow P(\overline{K}) \xrightarrow{\deg} \mathbf{Z} \rightarrow 0$$

where  $\deg$  is the degree map on  $\mathrm{Pic}(\overline{X})$ . Taking  $G_K$ -invariants of 2.2, we obtain the exact sequence

$$(2.3) \quad 0 \rightarrow A(K) \rightarrow P(K) \xrightarrow{\deg} \delta'\mathbf{Z} \rightarrow 0$$

where  $\delta'$  is the *period* of  $X$ . Equivalently,  $\delta'$  is the least positive degree of a divisor class in  $P(K) = \mathrm{Pic}(\overline{X})^{G_K}$ . The image of the map  $\mathrm{Div}(X) \rightarrow P(K)$  is denoted by  $\mathrm{Pic}(X)$ . Furthermore, it is known

that  $\text{Pic}(X) = H^1(X, \mathbb{G}_m)$ . Let  $\text{Br}(X) = H^2(X, \mathbb{G}_m)$  be the Brauer-Grothendieck group of  $X$ . By Lemma 2.2 in [Mil82], there is an exact sequence

$$(2.4) \quad 0 \rightarrow \text{Pic}(X) \rightarrow P(K) \rightarrow \text{Br}(K) \rightarrow \text{Br}(X) \rightarrow H^1(K, P) \rightarrow 0$$

The zero on the right-hand end follows from [Mil86, Cor.I.4.21].

### 3. THE MAIN THEOREM

In this section, we prove

**Theorem 3.1.** *Let  $X$  be a proper, smooth, geometrically connected curve over a non-archimedean local field  $K$  having finite residue field  $k$ , with index  $\delta$  and period  $\delta'$ . Let  $X^{\text{nr}} = X \otimes_K K^{\text{nr}}$  be the corresponding curve over  $K^{\text{nr}}$ , with index  $\delta^{\text{nr}}$  and period  $\delta^{\text{nr}'}$ . Let  $\text{Br}_0(X)$  denote the image of  $\text{Br}(K) \rightarrow \text{Br}(X)$ , and  $\text{Br}_{\text{nr}}(X)$  denote the kernel of the map  $\text{Br}(X) \rightarrow \text{Br}(X^{\text{nr}})$ . Let  $A$  be the Jacobian variety of  $X$  over  $K$ . Then there exists an exact sequence*

$$0 \rightarrow \text{Hom}(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \Phi_A(k) \rightarrow \mathbf{Z}/d\mathbf{Z} \rightarrow 0$$

where  $\Phi_A$  is the component group of  $A$ , and  $d = \delta'/\delta^{\text{nr}'}$ .

*Proof.* The short exact sequence

$$0 \rightarrow A \rightarrow P \rightarrow \mathbf{Z} \rightarrow 0$$

over  $K$  and  $K^{\text{nr}}$  gives rise, respectively, to the exact rows of the commutative diagram

$$\begin{array}{ccccccccc} P(K) & \longrightarrow & \mathbf{Z} & \longrightarrow & H^1(K, A) & \longrightarrow & H^1(K, P) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ P(K^{\text{nr}}) & \longrightarrow & \mathbf{Z} & \longrightarrow & H^1(K^{\text{nr}}, A) & \longrightarrow & H^1(K^{\text{nr}}, P) & \longrightarrow & 0 \end{array}$$

The columns in the diagram are induced by the inclusion  $K \subset K^{\text{nr}}$ . The image of the map  $P(K) \rightarrow \mathbf{Z}$  is, by definition,  $\delta'\mathbf{Z}$  while that of  $P(K^{\text{nr}}) \rightarrow \mathbf{Z}$  is  $\delta^{\text{nr}'}\mathbf{Z}$ . Thus we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}/\delta'\mathbf{Z} & \longrightarrow & H^1(K, A) & \longrightarrow & H^1(K, P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}/\delta^{\text{nr}'}\mathbf{Z} & \longrightarrow & H^1(K^{\text{nr}}, A) & \longrightarrow & H^1(K^{\text{nr}}, P) & \longrightarrow & 0 \end{array}$$

Since  $\delta^{\text{nr}'}$  divides  $\delta'$ , the leftmost vertical map is surjective. The kernel of this map is  $\mathbf{Z}/d\mathbf{Z}$  where  $d = \delta'/\delta^{\text{nr}'}$ . The middle vertical map has kernel  $H^1(K^{\text{nr}}/K, A(K^{\text{nr}})) \cong \text{TT}(A/K)$  by Theorem 2.1. The kernel

of the rightmost vertical map is  $H^1(K^{\text{nr}}/K, P(K^{\text{nr}}))$ . Snake Lemma then gives an exact sequence

$$(3.1) \quad 0 \rightarrow \mathbf{Z}/d\mathbf{Z} \rightarrow \text{TT}(A/K) \rightarrow H^1(K^{\text{nr}}/K, P(K^{\text{nr}})) \rightarrow 0$$

We now describe the right-most term in the exact sequence 3.1. Consider the commutative diagram

$$\begin{array}{ccccccccc} P(K) & \longrightarrow & \text{Br}(K) & \longrightarrow & \text{Br}(X) & \longrightarrow & H^1(K, P) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ P(K^{\text{nr}}) & \longrightarrow & \text{Br}(K^{\text{nr}}) & \longrightarrow & \text{Br}(X^{\text{nr}}) & \longrightarrow & H^1(K^{\text{nr}}, P) & \longrightarrow & 0 \end{array}$$

where the top and bottom row are obtained by applying 2.4 to  $X$  and  $X^{\text{nr}}$  respectively. Since  $\text{Br}(K^{\text{nr}}) = 0$  [Ser79], the above diagram reduces to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Br}_0(X) & \longrightarrow & \text{Br}(X) & \longrightarrow & H^1(K, P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \text{Br}(X^{\text{nr}}) & \longrightarrow & H^1(K^{\text{nr}}, P) & \longrightarrow & 0 \end{array}$$

Snake Lemma then yields the exact sequence

$$0 \rightarrow \text{Br}_0(X) \rightarrow \text{Br}_{\text{nr}}(X) \rightarrow H^1(K^{\text{nr}}/K, P(K^{\text{nr}})) \rightarrow 0$$

which yields the isomorphism

$$\text{Br}_{\text{nr}}(X)/\text{Br}_0(X) \cong H^1(K^{\text{nr}}/K, P(K^{\text{nr}}))$$

The exact sequence 3.1 can now be given as

$$(3.2) \quad 0 \rightarrow \mathbf{Z}/d\mathbf{Z} \rightarrow \text{TT}(A/K) \rightarrow \text{Br}_{\text{nr}}(X)/\text{Br}_0(X) \rightarrow 0$$

Dualizing this sequence, we obtain

$$(3.3) \quad 0 \rightarrow (\text{Br}_{\text{nr}}(X)/\text{Br}_0(X))^{\vee} \rightarrow \text{TT}(A/K)^{\vee} \rightarrow \mathbf{Z}/d\mathbf{Z}^{\vee} \rightarrow 0$$

Here  $M^{\vee} = \text{Hom}(M, \mathbf{Q}/\mathbf{Z})$  denotes the Pontryagin dual of  $M$ . By 2.1,  $\text{TT}(A/K)^{\vee} \cong \Phi_A(k)$ . On the other hand, the homomorphism  $\alpha : 1 \mapsto (\frac{1}{d} + \mathbf{Z})$  has order  $d$  and generates  $\text{Hom}(\mathbf{Z}/d\mathbf{Z}, \mathbf{Q}/\mathbf{Z})$ , so that  $\text{Hom}(\mathbf{Z}/d\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong \mathbf{Z}/d\mathbf{Z}$ . Thus 3.3 can be given as

$$0 \rightarrow \text{Hom}(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \Phi_A(k) \rightarrow \mathbf{Z}/d\mathbf{Z} \rightarrow 0$$

□

The following corollary is immediate

**Corollary 3.2.** *The quotient group  $\text{Br}_{\text{nr}}(X)/\text{Br}_0(X)$  is finite of order  $c_A/d$ .*

*Remark 3.3.* By [CTS13, Prop 2.1] (which applies to any proper, smooth, geometrically integral variety and not just a curve), the quotient group  $\text{Br}_{\text{nr}}(X)/\text{Br}_0(X)$  is finite. The order, however, does not seem to be recorded in the literature.

**Corollary 3.4.** *Suppose that  $\delta' = \delta^{\text{nr}'}$  (this happens, for example, if  $X$  has a  $K$ -rational point). Then there exists a canonical perfect pairing of finite abelian groups*

$$\text{Br}_{\text{nr}}(X)/\text{Br}_0(X) \times \Phi_A(k) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

*Proof.*  $\delta' = \delta^{\text{nr}'}$  implies that  $d = 1$ . The exact sequence 3.2 shows that, in this case, there exists a canonical isomorphism

$$\text{TT}(A/K) \cong \text{Br}_{\text{nr}}(X)/\text{Br}_0(X)$$

The pairing of the statement is then induced by the pairing of Corollary 2.2.  $\square$

*Remark 3.5.* Theorem 3.1 shows that  $c_A = \#\Phi_A(k)$  can be expressed as the product of  $\#(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X))$  and  $\delta'/\delta^{\text{nr}'}$ . On the other hand, Theorem 1.17 in [BL99] expresses  $\#\Phi_A(k)$  as the product of a term  $\#(\text{Ker}(\beta)/\text{Im}(\alpha))$  and  $\delta/(q\delta^{\text{nr}})$  (Note that the notations for  $\delta$  and  $\delta^{\text{nr}}$  are different in [BL99]). Both  $\text{Ker}(\beta)$  and  $\text{Im}(\alpha)$  are certain subgroups of the group of Weil divisors on  $X$  with support in the special fiber  $X_k$ . Letting  $g$  be the genus of  $X$ , we have that  $q = 1$  if  $\delta$  divides  $g - 1$ , and  $q = 2$  otherwise. As D. Lorenzini explained to us, by [Lic69, Thm 7.b],  $\delta/q = \delta'$  so that  $\delta/(q\delta^{\text{nr}}) = \delta'/\delta^{\text{nr}}$ . The commutative diagram in the proof of [Lic69, Theorem 3] then implies that since  $\text{Br}(K^{\text{nr}}) = 0$ , we have  $\delta^{\text{nr}} = \delta^{\text{nr}'}$ . Thus we have  $\delta'/\delta^{\text{nr}'} = \delta'/\delta^{\text{nr}} = \delta/(q\delta^{\text{nr}})$ . It follows that  $\#(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X)) = \#(\text{Ker}(\beta)/\text{Im}(\alpha))$ . Furthermore, comparing Cor 3.4 with Cor 1.12 in [BL99] yields the isomorphism

$$(\text{Ker}(\beta)/\text{Im}(\alpha)) \cong \text{Hom}(\text{Br}_{\text{nr}}(X)/\text{Br}_0(X), \mathbf{Q}/\mathbf{Z})$$

when  $d = 1$ .

*Remark 3.6.* The surjective map  $\Phi_A(k) \xrightarrow{\beta} \mathbf{Z}/d\mathbf{Z}$  in Theorem 3.1 can be made explicit. Consider the commutative diagram

$$\begin{array}{ccc} \Phi_A(k) & \xrightarrow{\beta} & \mathbf{Z}/d\mathbf{Z} \\ \alpha_1 \downarrow & & \uparrow \alpha_3 \\ \text{Hom}(\text{TT}(A/K), \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\alpha_2} & \text{Hom}(\mathbf{Z}/d\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \end{array}$$

Here  $\alpha_1$  is an isomorphism via the canonical perfect pairing of finite abelian groups  $\langle, \rangle : \Phi_A \times \text{TT}(A/K) \rightarrow \mathbf{Q}/\mathbf{Z}$  [McC86, (4.5)] which in



turn is induced by Grothendieck's pairing  $\Phi_A \times \Phi_A \rightarrow \mathbf{Q}/\mathbf{Z}$  [McC86, (2.1)]. The map  $\alpha_3$  is an isomorphism as explained in the proof of Theorem 3.1 above. Finally, the horizontal map  $\alpha_2$  is induced by the injective map  $\mathbf{Z}/d\mathbf{Z} \xrightarrow{\Delta} \mathrm{TT}(A/K)$  in the exact sequence 3.2. Note that  $\Delta$  is induced by the connecting homomorphism of the long exact sequence induced by the exact sequence  $0 \rightarrow A \rightarrow P \rightarrow \mathbf{Z} \rightarrow 0$  over  $K^{\mathrm{nr}}/K$ . If  $x \in \Phi_A(k)$ , then the composition  $(\alpha_2 \circ \alpha_1)(x)$  is the homomorphism  $\sigma : 1 \mapsto \langle x, \Delta(1) \rangle$ . Since  $\mathrm{Hom}(\mathbf{Z}/d\mathbf{Z}, \mathbf{Q}/\mathbf{Z})$  is generated by  $\alpha : 1 \mapsto \frac{1}{d} + \mathbf{Z}$ ,  $\sigma = m\alpha$  for some  $0 < m \leq d-1$ . Then  $\alpha_3(\sigma) = m$ , and we let  $\beta(x) = m$ . On the other hand, the injective map  $\mathrm{Hom}(\mathrm{Br}_{\mathrm{nr}}(X)/\mathrm{Br}_0(X), \mathbf{Q}/\mathbf{Z}) \rightarrow \Phi_A(k)$  in Theorem 3.1 is induced by  $\mathrm{TT}(A/K) \rightarrow H^1(K^{\mathrm{nr}}/K, P(K^{\mathrm{nr}})) \cong \mathrm{Br}_{\mathrm{nr}}(X)/\mathrm{Br}_0(X)$ . As shown in the proof of Theorem 3.1 above, the map  $\mathrm{TT}(A/K) \rightarrow H^1(K^{\mathrm{nr}}/K, P(K^{\mathrm{nr}}))$  is the restriction of the map  $H^1(K, A) \rightarrow H^1(K, P)$  to  $\mathrm{TT}(A/K)$ , and the latter map is induced by the surjective map  $A \rightarrow P$ . Finally, the isomorphism  $H^1(K^{\mathrm{nr}}/K, P(K^{\mathrm{nr}})) \cong \mathrm{Br}_{\mathrm{nr}}(X)/\mathrm{Br}_0(X)$  is induced by the map  $\mathrm{Br}(X) \xrightarrow{\psi} H^1(K, P)$  as in 2.4. The map  $\psi$  is described explicitly in [Mil82, Rem. 2.3].

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